Stress-dependent seismic anisotropy of shales

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ABSTRACT
A simple theory for the stress-dependent seismic anisotropy of shales can be obtained in terms of a second-rank tensor and a fourth-rank tensor that depend on the orientation distribution of contacts between clay platelets. The theory allows the normal and shear stiffness of the contact regions between clay particles to be obtained as a function of stress from measurements of seismic P- and S-wave velocities for shales. The ratio of the normal-to-shear compliance, $B_N/B_T$, of the contact regions between clay particles is found to be sensitive to the saturation state of the shale. Inversion of velocity measurements for fully saturated shales shows a low value of $B_N/B_T$ when compared with measurements on air-dry shales, consistent with the expected reduction in normal compliance in a fluid-saturated, low-permeability rock. For all shales considered, $B_N/B_T$ is found to be less than unity. The contacts between clay particles are therefore more compliant in shear than in compression.

THEORETICAL MODEL
It is assumed the stress dependence of the elastic properties of a shale is because of deformation of the contact regions between clay platelets. At high confining stress, these contacts are assumed to close so that the shale may be treated as a homogeneous anisotropic elastic medium with elastic stiffness tensor $C_{ijkl}$ and elastic compliance tensor $S_{ijkl}$. At intermediate values of the stress, it is assumed the contact regions between clay particles will be partially open (see Figure 1).

In Appendix A, the elastic compliance of the shale is

$$S_{ijkl} = S_{ijkl}^0 + \Delta S_{ijkl},$$

where the excess compliance, $\Delta S_{ijkl}$, because of the contact regions between clay platelets can be written as

$$\Delta S_{ijkl} = \frac{1}{4} (\delta_{ik} \alpha_{jl} + \delta_{jl} \alpha_{ik} + \delta_{jk} \alpha_{il} + \delta_{ij} \alpha_{lk}) + \beta_{ijkl}.$$

Here, $\alpha_{ij}$ is a second-rank tensor and $\beta_{ijkl}$ is a fourth-rank tensor defined by

$$\alpha_{ij} = \frac{1}{V} \sum_r B_T^{(r)} n_i^{(r)} n_j^{(r)} A^{(r)}$$

as illustrated in Figure 1 (Swan et al., 1989; Hornby et al., 1994; Schoenberg et al., 1996). The normal $n$ to the contacts between clay platelets varies from domain to domain, with a preferred orientation resulting from the depositional and stress history of the rock. Thus, the clay particles vary in orientation but are aligned locally (Swan et al., 1989; Hornby et al., 1994; Schoenberg et al., 1996).

A simple model of the stress-dependent elastic stiffness of shales having a domain structure is presented in which the elastic stiffness tensor is expressed in terms of the elastic stiffness of the shale at high confining stress and second- and fourth-rank tensors that depend on the orientation distribution of contacts between clay platelets. The theory allows the normal and shear stiffness of the contact regions between clay platelets to be obtained as a function of stress from measurements of the ultrasonic compressional- and shear-wave velocities for shales.
and

\[ \beta_{ijkl} = \frac{1}{V} \sum_{r} \left( B_N^{(r)} - B_T^{(r)} \right) n_i^{(r)} n_j^{(r)} n_k^{(r)} n_l^{(r)} A^{(r)}, \tag{4} \]

where \( B_N^{(r)} \) and \( B_T^{(r)} \) are the normal and shear compliances of the \( r \)th contact (see Figure 1 and Appendix A), \( n_i^{(r)} \) is the \( i \)th component of the normal to the contact, and \( A^{(r)} \) is the area of the contact plane. \( B_N^{(r)} \) characterizes the displacement discontinuity normal to the contact produced by a normal traction, while \( B_T^{(r)} \) characterizes the shear displacement discontinuity produced by a shear traction applied at the contact. \( B_T^{(r)} \) is assumed to be independent of the direction of the shear traction within the contact plane. Note that \( a_{ij} \) and \( \beta_{ijkl} \) are symmetric with respect to all rearrangements of the indices so that, for example, \( \beta_{i132} = \beta_{i213}, \beta_{i113} = \beta_{i311}, \) etc. If \( B_N = B_T \) for all contacts, \( \Delta S_{ijkl} \) is completely determined by the second-rank tensor \( a_{ij} \).

### INVERSION OF ULTRASONIC VELOCITY MEASUREMENTS

The components \( a_{ij} \) and \( \beta_{ijkl} \) of the second- and fourth-rank tensors introduced above may be obtained from equations (2) by inverting elastic stiffnesses obtained from measured P- and S-wave velocities. The shales considered are transversely isotropic (Hornby, 1994; Johnston and Christensen, 1993; Vernik, 1993). If the symmetry axis is chosen to lie along \( x_3 \), the nonvanishing \( \Delta S_{ijkl} \) obtained from equation (2) are given by equations (A-14)–(A-19) of Appendix A.

For the analysis of the experimental data, it is more convenient to use the conventional (two-subscript) condensed 6 × 6 matrix notation in which \( 11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3, 23 \rightarrow 4, 13 \rightarrow 5, \) and \( 12 \rightarrow 6 \) while factors 2 and 4 are introduced as follows (Nye, 1985):

\[ S_{ij} \rightarrow S_{pq} \quad \text{when both } p, q \text{ are } 1, 2, \text{ or } 3; \]
\[ 2S_{ijkl} \rightarrow S_{pq} \quad \text{when one of } p, q \text{ are } 4, 5, \text{ or } 6; \]
\[ 4S_{ijkl} \rightarrow S_{pq} \quad \text{when both } p, q \text{ are } 4, 5, \text{ or } 6. \]

Factors 2 and 4 are absent in the condensation of the stiffnesses tensor components so that

\[ C_{ijkl} \rightarrow C_{pq} \quad (i, j, k, l = 1, 2, 3; \quad p, q = 1, \ldots, 6). \]

Hornby (1994) measured ultrasonic wave velocities for two fully saturated shales of Jurassic age. Figure 2 shows the elastic stiffnesses calculated by Hornby (1994) from the measured ultrasonic wave velocities using a reference set of axes \( O_{x_1-x_2-x_3} \) with \( O_{x_3} \) perpendicular to the bedding plane. These values were inverted to obtain the elastic compliance components \( S_{ij} \) plotted in Figure 3.

Figure 4 shows the components of the second- and fourth-rank tensors \( a_{ij} \) and \( \beta_{ijkl} \) obtained from \( S_{ij} \) plotted in Figure 3 by using equation (2) and assuming the variation in the elastic stiffnesses of the clay particles with stress can be neglected. The elastic compliances \( S_{ij} \) corresponding to the case when the contacts are closed, were taken to be the values corresponding to the highest stress used in the experiment. These values are \( S_{11} = 29.3, S_{33} = 45.1, S_{13} = -13.1, S_{23} = 112.4, \) and \( S_{66} = 70.4 \text{ TPa}^{-1} \) for Figure 4a and \( S_{11} = 23.1, S_{33} = 39.9, S_{13} = -11.0, S_{23} = 97.1, \) and \( S_{66} = 53.2 \text{ TPa}^{-1} \) for Figure 4b. The components of the second- and fourth-rank tensors shown therefore correspond to the change in grain contacts between the lowest stress used (5 MPa) and the highest (80 MPa).

Figure 4 shows that all components of \( a_{ij} \) and \( \beta_{ijkl} \) except \( a_{33} \) and \( \beta_{3333} \) are small. It follows from equations (3) and (4) that most contacts are aligned normal to the symmetry axis \( O_{x_3} \). The opposite sign of \( a_{33} \) and \( \beta_{3333} \) implies that \( B_N/B_T < 1 \) (see equations (3) and (4)). The contacts between clay particles are therefore more compliant in shear than in compression.

If the contacts between clay particles are assumed to be perpendicular to the symmetry axis, a comparison of equations (3) and (4) with the values of \( a_{33} \) and \( \beta_{3333} \) plotted in Figure 4 allows the ratio \( B_N/B_T \) of normal-to-shear compliance of the contact regions to be estimated for each value of the confining stress. Between 5 and 60 MPa confining stress, \( B_N/B_T \) takes values in the range \( 0.26 < B_N/B_T < 0.30 \) for Figure 4a, with an average value of 0.29. For Figure 4b the values lie in the range \( 0.33 < B_N/B_T < 0.41 \), with an average value of 0.37.

It is interesting to compare these results with those obtained for shales under air-dry conditions. Johnston and Christensen (1993) calculate the elastic stiffnesses for air-dry shales from the Millboro and Brailleur members of the Devonian–Mississippian Chattanooga Formation from their ultrasonic velocity measurements (see Figure 5). These values were inverted to obtain the elastic compliance components \( S_{ij} \) shown in Figure 6. The components of the second- and fourth-rank tensors \( a_{ij} \) and \( \beta_{ijkl} \) were then obtained by using equation (2). The results are shown in Figure 7.

\( B_N/B_T \) was estimated for these shales from the values of \( a_{33} \) and \( \beta_{3333} \) plotted in Figure 7 by assuming the grain contacts to be parallel to the bedding plane. Between 20 and 100 MPa confining stress, \( B_N/B_T \) takes values in the range \( 0.47 < B_N/B_T < 0.58 \) for the Millboro sample, with an average value of 0.52. For the Brailleur sample the values lie in the range \( 0.54 < B_N/B_T < 0.63 \), with an average value of 0.58. The contacts between clay particles are again more compliant in shear than in compression, although the ratio is increased compared to that for fluid-saturated shales. This difference is consistent with the expected reduction in normal compliance in a fluid-saturated, low-permeability rock.

Vernik (1993) measured ultrasonic velocities for an air-dry mature, kerogen-rich shale; the elastic stiffnesses and compliances obtained from these measurements are shown in
Figure 8. This sample contained bedding-parallel microcracks caused by the process of hydrocarbon generation. Figure 9 shows the components of the second- and fourth-rank tensors $\alpha_{ij}$ and $\beta_{ijkl}$ obtained from $S_{ij}$ plotted in Figure 8 by using equation (2). The ratio $B_N/B_T$ was estimated from the values of $\alpha_{33}$ and $\beta_{333}$ plotted by assuming the grain contacts and microcracks to be parallel to the bedding plane. $B_N/B_T$ takes values between 5 and 30 MPa confining stress in the range $0.6 < B_N/B_T < 0.8$, with an average value of 0.68.

**CONCLUSION**

A simple theory has been presented in which the stress-dependent seismic anisotropy of shales is obtained in terms of components $\alpha_{ij}$ and $\beta_{ijkl}$ of a second-rank and a fourth-rank tensor that depend on the orientation distribution of contacts between clay platelets. The theory allows the normal and shear stiffnesses of the contact regions between clay platelets to be obtained as a function of stress from measurements of seismic $P$- and $S$-wave velocities for shales. For the shales considered, all components of $\alpha_{ij}$ and $\beta_{ijkl}$ except $\alpha_{33}$ and $\beta_{333}$ are small. It follows, from equations (3) and (4) that most contacts are aligned normal to the symmetry axis. The opposite sign of $\alpha_{33}$ and $\beta_{333}$ implies that $B_N/B_T < 1$ [see equations (3) and (4)]. The contacts between clay particles are therefore more compliant in shear than in compression. The ratio $B_N/B_T$ of the normal-to-shear compliance of the contact regions between clay particles is sensitive to the saturation state of the shale. Inversion of measurements of Hornby (1994) for fully saturated samples shows a reduced value of $B_N/B_T$ compared to air-dry shales, consistent with the expected reduction in normal compliance in a fluid-saturated, low-permeability rock.
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REFERENCES


It is assumed that the stress dependence of the elastic properties of a shale is because of deformation of the contact regions between clay platelets (see Figure 1). Consider a volume $V$ containing $N$ contact regions (with surfaces $S^r$, $r = 1, \ldots, N$). The strain tensor $\varepsilon_{ij}$ is defined in terms of the displacement vector $u_i$ by

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (A-1)$$

Using Gauss’s theorem, the average strain tensor in the solid phase in volume $V$ may be written (Hill, 1963) as

$$\frac{1}{V} \int \varepsilon_{ij} \, dV = \bar{\varepsilon}_{ij} + \frac{1}{2V} \sum_r \int_{S^r} \left( u_i n_j + u_j n_i \right) \, dS, \quad (A-2)$$

where $n_i$ is the $i$th component of the outward normal to the solid, $V_i$ is the volume of the solid phase, and $\bar{\varepsilon}_{ij}$ is defined by

$$\bar{\varepsilon}_{ij} = \frac{1}{2V} \int_{S_e} \left( u_i n_j + u_j n_i \right) \, dS \quad (A-3)$$

where $S_e$ is the solid portion of the exterior boundary.

The macroscopic strains $\varepsilon_{ij}$ are related to the macroscopic stress components $\sigma_{ij}$ [defined by $\sigma_{ij} = (1/V) \int_{V_i} \sigma_{ij} \, dV$] by the effective compliance tensor $S_{ijkl}$:

$$\varepsilon_{ij} = S_{ijkl} \sigma_{kl}. \quad (A-4)$$

The strain $\varepsilon_{ij}$ in the solid phase is given by $\varepsilon_{ij} = S_{ijkl}^{0} \sigma_{kl}$, where $S_{ijkl}^{0}$ is the compliance tensor of the solid. For thin contact regions, the integral on the right side of equation (A-2) may be evaluated by taking the integration over the area $A^r$ of the contact and replacing displacements by displacement jumps across $A^r$ while putting $V = V_i$. This gives

$$\bar{\varepsilon}_{ij} = S_{ijkl}^{0} \bar{\sigma}_{kl} + \frac{1}{2V} \sum_{r=1}^{N} \int_{A^r} \left( \left[ u_i n_j^0 \right]^r + \left[ u_j n_i^0 \right]^r \right) \, dA$$

$$= (S_{ijkl}^{0} + \Delta S_{ijkl}) \bar{\sigma}_{kl} \quad (A-5)$$

(Sayers and Kachanov, 1995). Here $[u_i]$ denotes the $i$th component of the displacement discontinuity at the contact and $n_i$ is the $i$th component of the unit normal to the contact. The first term in equation (A-5) is from the deformation of the anisotropic matrix ($S_{ijkl}^{0}$ are the matrix compliances), and the second term reflects the additional compliance because of the contact regions between clay platelets.

It is assumed that the contacts are flat with unit normals $n_i^0$ that are constant along each $A^r$. The value $n_i^0$ may then be taken out of the integral on the right side of equation (A-5), and the extra compliance from contacts may be expressed in terms of the average displacement discontinuities $\left[ u_i \right]^r = (1/A^r) \int_{A^r} \left[ u_i \right] \, dA$. The problem is reduced to finding $\left[ u_i \right]^r$ in terms of the applied stresses.

It is assumed that the stress interactions between contacts may be neglected so each contact region may be considered as subjected to the average stress field $\bar{\sigma}_{ij}$. Introducing a symmetric second-rank compliance tensor $B$ (Kachanov, 1992) that expresses the vector of average displacement discontinuity in terms of the uniform traction, with $i$th component $t_i$, applied at the faces of the contact region between clay platelets gives, for the $i$th contact,

$$\left[ u_i \right]^r = B_i^{(r)} t_j, \quad (A-6)$$

where $t_j$ is given by

$$t_j = \bar{\sigma}_{jk} n_k. \quad (A-7)$$

The change in compliance because of the presence of the contacts is then obtained from equation (A-5) as

$$\Delta S_{ijkl} \bar{\sigma}_{kl} \equiv \frac{1}{2V} \sum_r \int_{A^r} \left[ u_i n_j + u_j n_i \right] \, dA$$

$$= \frac{1}{2V} \sum_r \left( B_{ij}^{(r)} \bar{\sigma}_{pq} n_q^0 n_i^0 + B_{ji}^{(r)} \bar{\sigma}_{pq} n_p^0 n_j^0 \right) A^r. \quad (A-8)$$

The shear compliance of the contact regions is assumed independent of direction in the plane of the contact so that

$$B_{ij} = B_N n_i n_j + B_T (\delta_{ij} - n_i n_j). \quad (A-9)$$

It follows that

$$\Delta S_{ijkl} \bar{\sigma}_{kl} = \frac{1}{2V} \sum_r \left[ B_{ij}^{(r)} \left( \bar{\sigma}_{pq} n_q^0 n_i^0 + \bar{\sigma}_{pq} n_p^0 n_j^0 \right) + 2 \left( B_N - B_T^{(r)} \right) \bar{\sigma}_{pq} n_i^0 n_j^0 n_q^0 \right] A^r. \quad (A-10)$$

To obtain the individual components $\Delta S_{ijkl}$, consider a test stress

$$\bar{\sigma}_{ij} = \frac{\sigma}{2} (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}). \quad (A-11)$$
This gives

\[ \Delta S_{ijkl} = \frac{1}{V} \sum_r \left[ \frac{B_{r}^{(r)}}{4} \left( \delta_{ik} n_{j}^{(r)} n_{l}^{(r)} + \delta_{il} n_{j}^{(r)} n_{k}^{(r)} ight) 
+ \delta_{jk} n_{i}^{(r)} n_{l}^{(r)} + \delta_{jl} n_{i}^{(r)} n_{k}^{(r)} \right] A^{(r)} \]  

(A-12)

upon substituting in equation (A-10) (Sayers and Kachanov, 1995).

Defining a second-rank tensor \( a_{ij} \) and a fourth-rank tensor \( \beta_{ijkl} \) by equations (3) and (4) in the text, it then follows from equation (A-12) that

\[ \Delta S_{ijkl} = \frac{1}{4} \left( \delta_{ik} a_{jl} + \delta_{il} a_{jk} + \delta_{jk} a_{il} + \delta_{jl} a_{ik} \right) + \beta_{ijkl}. \]

(A-13)

This equation quantifies the excess compliance from the contacts between clay particles in terms of the second- and fourth-rank tensors \( a_{ij} \) and \( \beta_{ijkl} \).

For a general transversely isotropic orientation distribution of contacts between clay platelets, \( a_{11} = a_{22}, \beta_{1111} = \beta_{2222}, \beta_{212} = \beta_{132} = \beta_{1111}/3 \), and equation (A-13) reduces to

\[ \Delta S_{1111} = \Delta S_{2222} = a_{11} + \beta_{1111}, \]

\[ \Delta S_{3333} = a_{33} + \beta_{3333}, \]

\[ \Delta S_{1212} = a_{11}/2 + \beta_{1111}/3, \]

\[ \Delta S_{2323} = \Delta S_{3131} = (a_{11} + a_{33})/4 + \beta_{1133}, \]

\[ \Delta S_{1122} = \beta_{1111}/3, \]

\[ \Delta S_{2233} = \Delta S_{3311} = \beta_{1133}. \]

(A-14) \hspace{1cm} (A-15) \hspace{1cm} (A-16) \hspace{1cm} (A-17) \hspace{1cm} (A-18) \hspace{1cm} (A-19)